SOME PROBLEMS ON CONTINUOUS FUNCTION OF A SINGLE REAL VARIABLE AND OF REAL VALUED

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01. Let $f: [0,1] \rightarrow R$ be a differential function such that $f(0) = 0$ and $f(1) = 1$. Show that there exists $a, b \in (0,1)$ with $a \neq b$ such that $\frac{1}{f^{/}(a)} + \frac{1}{f^{/}(b)} = 2$.

02. If $f: [-1,1] \rightarrow R$ is continuous function, then show that there exists $c \in [-1,1]$ such that $|f(c)| = \frac{1}{4}(|f(-1)| + 2|f(0)| + |f(1)|)$.

03. Statement: "There exists a continuous function $f: [1,2] \rightarrow R$ which is differential on $(0,1)$ but not differential at the points 0 and 1." Justify the statement.

04. Let $f: A \rightarrow R$ and $g: B \rightarrow R$ where $f(A) \subset B$. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then prove that $g_o f$ is continuous at c.

05. Let $f: A \rightarrow R$ and $g: B \rightarrow R$ where $f(A) \subset B$. If f is continuous on $A$and g is continuous on B, then prove that $g_o f$ is continuous on A.

06. A function $f: R \rightarrow R$ is continuous on R and $f(x) = 0, \forall x \in Q$. Prove that $f(x) = 0$ for all $x \in R$.

07. A function $f: R \rightarrow R$ satisfies the condition $f(x + y) = f(x)f(y)$ for all $x, y \in R$. If f is continuous at $x = 0$, prove that the function f is continuous on R.

08. A function $f: R \rightarrow R$ satisfies the condition $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. If f is continuous at one point $c \in R$, prove that the function f is continuous at every point in R.

09. A function $f: R \rightarrow R$ is continuous on R and $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all $x, y \in R$. Prove that $f(x) = ax + b, (a, b \in R)$ for all $x, y \in R$.

10. Let $f: (-1,1) \rightarrow R$ be continuous at $x = 0$. If $f(x) = f(x^2)$ for all $x, y \in (-1,1)$, prove that $f(x) = f(0)$ for all $x, y \in (-1,1)$.

11. A function $f: R \rightarrow R$ is continuous on R and $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. If $f(1) = k$ prove that $f(x) = kx$ for all $x \in R$. [k is a real constant]

12. A function $f: R \rightarrow R$ is continuous on R and $f(x + y) = f(x)f(y)$ for all $x, y \in R$. Prove that either $f(x) = 0$ or $f(x) = a^x$ for all $x \in R$, where a is some positive real number.

13. Let $f: [a, b] \rightarrow R$ be strictly increasing. Show that the inverse function f^{-1} exists and strictly increasing on $[\alpha, \beta]$ where $\alpha = f(a)$ and $\beta = f(b)$. If further, the function f is continuous on $[a, b]$ show that f^{-1} is also continuous on $[\alpha, \beta]$.

14. Prove that the image of a closed interval under a continuous function $f: R \rightarrow R$ is a closed interval.

15. Let $f: [a, b] \rightarrow [a, b]$ be a continuous function. Prove that there exists at least one point $c \in [a, b]$ such that $f(c) = c$.

16. Let $f: [a, b] \rightarrow R$ be a continuous function. Prove that there exists a point $c \in (a, b)$ such that $f(c) = \frac{f(a)+f(b)}{2}$.

17. Sum of two discontinuous functions –

a. is a discontinuous function.

b. may not be a discontinuous function.

Which of the above two statements is true? Support your answer with appropriate reason.

18. Let $f: [a, b] \to R$ be a continuous function and it assumes each value between $f(a)$ and $f(b)$ just once. Prove that the function f is strictly monotone on $[a, b]$.

19. A function $f: [a, b] \to R$ is continuous on $[a, b]$ and $x_1, x_2, x_3 \in [a, b]$. Prove that there exists a point $c \in [a, b]$ such that $f(c) = \frac{f(x_1)+f(x_2)+f(x_3)}{3}$.

20. A function $f: [a, b] \to R$ is continuous on $[a, b]$ and $x_1, x_2, x_3, \dots \dots x_n \in [a, b]$. Prove that there exists a point $c \in [a, b]$ such that $f(c) = \frac{f(x_1)+f(x_2)+f(x_3)+\cdots\dots\dots+f(x_n)}{n}$.

21. Let $f: [a, b] \to R$, $g: [a, b] \to R$ be continuous functions on $[a, b]$ having the same range $[0,1]$. Prove that there exists a point $c \in [a, b]$ such that $f(c) = g(c)$.

22. Prove that if $f: R \to R$ and $g: R \to R$ are continuous functions and $f(a) < g(a)$ at some point $a \in R$, then a has a neighbourhood where $f(x) < g(x)$ for all x belongs to the neighbourhood.

23. Does the continuity of the function $g(x) = f(x^2)$ implies the continuity of the function $f(x)$? Justify your answer with proper arguments.

24. Assume that the function $g(x) = \lim_{t\to x} f(t)$ exists for all $x \in R$. Prove that $g(x)$ is a continuous function.

25. Suppose that $f: R \to R$ is continuous and $f(n.a) \to 0$ for all $a > 0$. Prove that $\lim_{x\to\infty} f = 0$.

26. A function $f: R \to R$ satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. If the function f is continuous at some point $c \in R$ prove that the function f is uniformly continuous on R.

27. Let A be a non – empty subset of R. A function $f: R \to R$ is defined by $f_A(x) = inf\{|x - a|: a \in A\}$. Prove that the function f_A is uniformly continuous on R.

28. Let $c \in R$ and a function $f: R \to R$ is continuous at c. If for every positive δ there is appoint $y \in (c - \delta, c + \delta)$ such that $f(y) = 0$, prove that $f(c) = 0$.

29. Let $f: R \to R$ be continuous on R and let $c \in R$ such that $f(c) > \mu$. Prove that there exists a neighbourhood U of c such that $f(x) > \mu$ for all $x \in U$.

30. Let a function $f: R \to R$ be continuous on R. Prove that the set $Z(f) = \{x \in R: f(x) = 0\}$ is a closed set in R. Give an example of a function f continuous on R such that –

a. $Z(f)$ is a bounded enumerable set

b. $Z(f)$ is an unbounded enumerable set

31. Let a function $f: R \to R$ be continuous on R. A point $c \in R$ is said to be a fixed point of f if $f(c) = c$ holds. Prove that the set of all fixed points of the function f is a closed set.

32. Let $I = [a, b]$ be a closed and bounded interval and a function $f: I \to R$ be continuous on I and $f(x) > 0$ for all $x \in I$. Prove that there exists a positive number α such that $f(x) \geq \alpha$ for all $x \in I$.

33. A function $f: [0,1] \to R$ is continuous on $[0,1]$ and the function assumes only rational values on $[0,1]$. Prove that the function f is a constant function.

34. Let $f: [0,2] \to R$ be continuous and $f(0) = f(2)$. Prove that there exists at least a point $c \in [0,1]$ such that $f(c) = f(c + 1)$.

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