July, 2020

01. Prove the following statements for all $n \in N$ (set of all natural numbers) by Principle of Mathematical Induction –

a. $1 + 2 + 3 + \cdots \dots \dots \dots \dots \dots \dots \dots . + n = \frac{n(n+1)}{2}$

b. $1^2 + 2^2 + 3^2 + \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots . + n^2 = \frac{n(n+1)(2n+1)}{6}$

c. $1^3 + 2^3 + 3^3 + \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

02. By Principle of Mathematical Induction prove that –

a. $\sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}$ for all natural numbers $n \geq 2$

b. $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \dots \dots \dots \dots \dots \dots \dots \dots \left(1 - \frac{1}{n^2}\right) = \frac{n(n+1)}{2n}$ for all natural numbers $n \geq 2$

03. Problems on divisibility:

i. Prove by Principle of Mathematical Induction that $2^{2n} - 1$ is divisible by 3.

ii. Prove by Principle of Mathematical Induction that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all integers $n \geq 0$.

04. Prove that $2n + 1 < 2^n$ for all natural numbers $n \geq 3$ by Principle of Mathematical Induction.

05. Use the Principle of Mathematical Induction to verify that, for n any positive integer, $6^n - 1$ is divisible by 5.

06. Use Principle of Mathematical Induction to verify that, for n any positive integer, the sum of the squares of the first $2n$ positive integers is given by the formula –

$1^2 + 2^2 + 3^2 + \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . . + (2n)^2 = \frac{2n(2n+1)(4n+1)}{6} = \frac{n(2n+1)(4n+1)}{3}$.

07. Let $x \neq 0$ be a real number such that $x + x^{-1} \in Z$. Prove that for all $n \in Z$, $x^n + x^{-n} \in Z$.

08. Let the polar representation of a non – zero complex number $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = arg(z)$. Prove that $z^n = r^n(\cos(n\theta) + i\sin(n\theta)), \forall n \in Z$ by Principle of Mathematical Induction.

RECURSION

09. Let $a_0 = 0$ and $a_n = 2a_{n-1} + n$ whenever $n \geq 1$. Show that $a_n = 2^{n+1} - n - 2$.

10. Let $a_0 = 0, a_1 = 1$ and $a_{n+2} = \frac{1}{4}(a_{n+1}^2 + a_n + 2)$ whenever $n \geq 0$. Show that $0 \leq a_n \leq 1$ for integers $n \geq 0$.

11. Let $a_0 = 1$ and $a_n = \sum_{i=0}^{n-1} a_i$. Prove that for all $n \geq 1$, we have $a_n = 2^{n-1}$.

12. Define the sequence $a = \{a_n\}$ as follows –

$a_1 = 2, a_n = 5a_{n-1}, \forall n \geq 2$. Write down the first four terms of the sequence and by Principle of Mathematical Induction prove that $a_n = 2.5^{n-1}$ for all natural numbers.

13. The distributive law from algebra says that, for all real numbers c, a_1, a_2 we have $c(a_1 + a_2) = ca_1 + ca_2$. Use this law and Principle of Mathematical Induction prove that for all natural numbers $n \geq 2$, if $c, a_1, a_2, \dots \dots \dots \dots . . , a_n$ are any real numbers, then $c.(a_1 + a_2 + a_3 + \cdots \dots \dots \dots \dots . . + a_n) = ca_1 + c.a_2 + c.a_3 + \cdots \dots \dots \dots \dots \dots \dots + c.a_n$.

14. Prove by Principle of Mathematical Induction that for all natural numbers n, $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots . + \sin(\alpha+(n-1)\beta) = \frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$.

15. Consider the sequence of real numbers defined by the recursive relations –

$x_1 = 1$, and $x_n = \sqrt{(1+2x_{n-1})}$, for natural numbers $n \geq 2$. Use Principle of Mathematical Induction to show that $x_n < 4$, for all natural numbers $n \geq 1$.

16. Let $p_o = 1, p_1 = \cos\theta$ (θ is some fixed constant) and $p_{n+1} = 2p_1p_n - p_{n-1}, \forall n \geq 1$. Use an extended Principle of Mathematical Induction to show that $p_n = \cos(n\theta)$ for all $n \geq 0$.

17. Consider the famous Fibonacci sequence $x = \{x_n\}$, defined by the relations $x_1 = 1, x_2 = 1$ and $x_n = x_{n-1} + x_{n-2}$, for all natural numbers $n \geq 3$.

a. Compute x_{20}.

b. Use an extended Principle of Mathematical Induction to prove that $x_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ for all $n \geq 1$.

18. Let $a_1 = 1, a_2 = 8$ and $a_n = a_{n-1} + 2.a_{n-2}$ whenever $n \geq 2$. Using Principle of Mathematical Induction to show that for all $n \geq 1, a_n = 3.2^{n-1} + 2.(-1)^n$.

19. Define the matrix A below and show the formula for A^n:

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$, where (f_j) are the Fibonacci numbers.

20. Prove the Binomial theorem by Principle of Mathematical Induction. This states that for all $n \geq 1$, $(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$.

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