বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2021

(Under CBCS Pattern)

Semester - V

Subject : MATHEMATICS

Paper : DSE 1 - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

[LINEAR PROGRAMMING]

Group-A

1. Answer any ***four*** questions : 12×4=48

(i) (a) Prove that the set of all convex combinations of a finite number of linearly independent vectors $X_1, \ldots, X_k$ is a convex set.

(b) Solve the following L.P.P by Big-M method

Maximize $z = x_1 - 2x_2 + 3x_3$

Subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1,\ x_2,\ x_3 \geq 0$ 4+8

(ii) (a) Use graphical method, show that the following L.P.P have no feasible solution

Maximize $z = 3x + 4y$

Subject to $x - y \leq 1$

$-x + y \leq 0$

$x, y \geq 0$

(b) Examine whether the set is convex or not

$X = \{(x_1, x_2), x_1 \geq 2,\ x_2 \leq 3,\ x_1,\ x_2 \geq 0\}$ 6+6

(iii) (a) Solve the following L.P.P using simplex method

Maximize $z = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \geq -2$

$-2x_1 + x_2 - 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1,\ x_2,\ x_3 \geq 0$

(b) Write the dual of the following L.P.P.

Maximize $z = x_1 + 2x_2$

Subject to $2x_1 + 3x_2 \geq 4$

$3x_1 + 4x_2 = 5$

$x_1 \geq 0$ and x_2 is unrestricted. 8+4

(iv) (a) Prove that the set of all feasible solutions to a linear programming problem is a convex set.

(b) Use two phase simplex method to solve the following L.P.P

Maximize $z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 40$

$x_1 + x_2 \leq 24$

$2x_1 + 3x_2 \leq 60$

$x_1,\ x_2 \geq 0$ 4+8

(v) (a) Prove that the dual of the dual is primal.

(b) Solve the following transportation problem using North-West corner method :

Destination

	X	Y	Z	W	Supply
A	5	4	6	14	15
Origin B	2	9	8	6	4
C	6	11	7	13	8
Demand	9	7	5	6	

5+7

(vi) (a) Prove that the number of basis variables in a transportation problem with m origin and n destinations is almost $m + n - 1$.

(b) Solve the following travelling salesman problem :

	A	B	C	D	E
A	∞	2	4	7	1
B	5	∞	2	8	2
C	7	6	∞	4	6
D	10	3	5	∞	4
E	1	2	2	8	∞

(vii) (a) Find the optimal assignment and the optimal assignment cost from the following cost matrix :

	M_1	M_2	M_3	M_4	M_5
I	9	8	7	6	4
II	5	7	5	6	8
III	8	7	6	3	5
IV	8	5	4	9	3
V	6	7	6	8	5

(b) Prove that if we add a fixed number to each element of a pay-of matrix, the optimal strategies remain unchanged but the value of the game is increased by that number.

7+5

(viii) (a) Solve graphically the game whose pay-off matrix is

Player B

$$\text{Player } A\begin{pmatrix} 2 & 2 & 3 & -1 \\ 4 & 3 & 2 & 6 \end{pmatrix}$$

(b) Use dominance property to reduce the pay-off matrix given by

Player B

$$\text{Plaer } A\begin{pmatrix} 3 & -1 & 1 & 2 \\ -2 & 3 & 2 & 6 \\ 2 & -2 & -1 & 1 \end{pmatrix}$$

into a 2×2 matrix and find the mixed strategies for A and B. Also find the value of the game. 6+6

Group-B

2. Answer any ***six*** questions : 2×6=12

(i) Find the extreme point, if any, of the set

$$S = \{(x, y) : |x| \le 1, |y| \le 1\}.$$

(ii) State the fundamental theorem of linear programming.

(iii) Define zero sum game.

(iv) Solve the games with the following payoff matrix $\begin{pmatrix} 6 & -3 \\ -3 & 0 \end{pmatrix}$.

(v) Prove that the solution of a transportation problem is never unbounded.

(vi) Define saddle point of a game.

(vii) Define basic feasible solution of an L.P.P.

(viii) For what values of a, the game with the following payoff matrix is strictly determinable?

	I	II	II
I	a	5	2
II	–1	a	–8
III	–2	3	a

(ix) Put the following problem in standard from

Maximize $z = 3x_1 - 4x_2 - x_3$

Subject to $x_1 + 3x_2 - 4x_3 \leq 12$

$2x_1 - x_2 + x_3 \leq 20$

$x_1 - 4x_2 - 5x_3 \geq 5$

$x_1 \geq 0,\ x_2$ and

x_3 are unrestricted in sign.

(x) In a game with the 2×2 payof matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a < d < b < c$, show that there is no saddle point.

OR

[POINT SET TOPOLOGY]

Group-A

1. Answer any ***four*** questions : 12×4=48

(a) (i) Check whether the set $A=\{f:\{0,1\}\to\mathbb{Z}^+ \mid f$ *is a function and* Z^+ *denote the set of all positive integers*$\}$ is countable or not.

(ii) Prove $(\mathbb{R}, \tau_u)$, where τ_u denotes the usual topology on $\mathbb{R}$, has a countable base.

(iii) Show that $S=\{(-\infty, a) \mid a\in\mathbb{R}\}\cup\{(b, \infty) \mid b\in\mathbb{R}\}$ is a subbase for the lower limit topology τ_l on $\mathbb{R}$.

(iv) Prove that the union of collection of connected sets having a point in common is connected. 3+3+3+3=12

(b) (i) Show that every well-ordered set has the least upper bound property.

(ii) Prove that in a topological space (X, τ), $bd(A)=\overline{A}\cap\overline{A^C}$ where A is a non-empty subset of X.

(iii) Let $(X, \tau),(Y, \tau')$ be two topological spaces and $b\in Y$. Then prove that X and $X\times\{b\}$ are homeomorphic. Hence prove that the product space $X\times Y$ is connected, if X and Y are connected.

(iv) Prove that $\mathbb{R}$ with respect to the usual topology τ_u is not compact.

2+2+(3+3)+2=12

(c) (i) Let A_1 and A_2 be disjoint sets, well-ordered by $<_1$ and $<_2$, respectively. Define an order relation on $A_1\cup A_2$ by letting $a<b$ either if $a, b\in A_1$ and $a<_1 b$, or if $a, b\in A_2$ and $a<_2 b$, or if $a\in A_1$ and $b\in A_2$. Show that this is a well-ordering.

(ii) Consider a family of non-empty sets $\{X_\alpha \mid \alpha\in\Lambda\}$ where Λ is an infinite set. Prove that the box topology is finer than the product topology on $\Pi_{\alpha\in\Lambda}X_\alpha$.

(iii) Define totally bounded metric space with an example. 5+5+2=12

(d) (i) Given two points (x_0, y_0) and (x_1, y_1) in $\mathbb{R}^2$, define $(x_0, y_0) < (x_1, y_1)$ if $x_0 < x_1$ and $y_0 \le y_1$. Show that the curve $y = x^3$ is a maximal simply ordered subset of $\mathbb{R}^2$.

(ii) Let *X* be a non-empty set and consider the following metric on X :

$$d(x,y) = \begin{cases} 1 & if\ x \ne y \\ 0 & if\ x = y \end{cases}$$

Prove that the metric topolgocy on *X* induced by *d* is the discrete topology on *X*.

(iii) Define open map, Let *X* be the subspace [0, 1] ∪ [2,3] of the topological space ($\mathbb{R}$, τ_u) and *Y* be the subspace [0, 2] of the topological space ($\mathbb{R}$, τ_u) where τ_u denotes the usual topology on $\mathbb{R}$). Is the map $p : X \to Y$ an open map where *p* is defined as follows :

$$p(x) = \begin{cases} x, & if\ x \in [0,1] \\ x-1, & if\ x \in [2, 3] \end{cases}?$$

Justify your answer.

(iv) Prove that every closed subspace of a compact space is compact.

3+2+(1+2)+4=12

(e) (i) Using the Axiom of choice show that if $f : A \to B$ is a surjective function, then f has a right inverse $h : B \to A$.

(ii) Deine a quotient map between two topological spaces. Then deine the quotient topology induced by the function *p* on a set *A* where (*X*, τ) is a topologcial space and $p : X \to A$ is a surjective map.

(iii) Consider ($\mathbb{R}$, τ_u) where τ_u denote the usual topology on $\mathbb{R}$. Define a surjective map *p* from $\mathbb{R}$ onto a three-element set *A* = {*a, b, c*} defined by

$$p(x) = \begin{cases} a, & if\ x > 0 \\ b, & if\ x < 0 \\ c, & if\ x = 0 \end{cases}$$

Compute the quotient topology on *A* induced by the surjective function *p*.

(iv) Deine finite complement (co-finite) topology on a set X. Then show that an infinite set X is always connected with respect to the finite complement topology.

(v) Give example (with justification) of a locally connected subspace which is not a connected subspace in $(\mathbb{R}, \tau_u)$ where τ_u denotes the usual topology on $\mathbb{R}$.
2+(1+1)+2+(1+2)+3=12

(f) (i) Consider the strict partial orderd set $(\mathbb{R}^2, <)$ where for $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$, $(x_0, y_0) < (x_1, y_1)$ if any only if $x_0 = y_1$ and $x_0 < x_1$. Find a maximal simple ordered subset of $(\mathbb{R}^2, <)$.

(ii) Let (X, τ) and (X, τ') be two topological spaces, $B \subseteq \tau'$ be a base for the topology τ' on Y and $f : X \to Y$ be a mapping. Then prove that f is continuous if and only $f^{-1}(B) \in \tau$ for all $B \in \mathrm{B}$.

(iii) Let τ and τ' be two topologies on a set X such that $\tau \subseteq \tau'$. Prove that the connectedness of (X, τ') implies connectedness of (X, τ). By exhibiting an example (with justification) show that the connectedness of (X, τ) need not always imply the connectedness of (X, τ'). 3+3+(3+3)=12

(g) (i) State Schroeder-Bernstein Theorem.

(ii) Give example of a continuous bijective function between two topological spaces which fails to be a homeomorphism.

(iii) Consider a family of topological spaces $\{(X_a, \tau_a) \mid \alpha \in \Lambda\}$ where Λ is an infinite set. Let $A_\alpha \subseteq X_\alpha$ for each $\alpha \in A$. Then prove that $\overline{\prod_{\alpha\in\Lambda} A_\alpha} = \prod_{\alpha\in\Lambda} \overline{A_\alpha}$ holds in the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

(iv) Show that finite union of compact subspaces in a topological space in compact again.
2+3+4+3=12

(h) (i) Show that in a well-ordered set, every element except the largest (if exists) has an immediate successor.

(ii) Give an example (with justification) of a function on a topological space which is continuous precisely at one point.

(iii) Prove that the image of a compact space under a continuous map is compact.

(iv) Let τ_1 and τ_2 be two topologies on a set X such that $\tau_1 \subseteq \tau_2$. Does the compactness of (X, τ_1) imply the compactness of (X, τ_2)? Justify your answer explicitly. 3+3+3+3=12

Group-B

2. Answer any ***six*** questions : 2×6=12

(a) Give an example (with justiication) of a countably infinite set.

(b) State axiom of choice.

(c) Prove that $(0, 1) \subseteq \mathbb{R}$ is not well ordered.

(d) Show that any uncountable set has greater cardinality than $\mathbb{N}$.

(e) Prove that the usual topology is coarser than the lower limit topology on $\mathbb{R}$.

(f) Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$ with the usual topology. Is $A = \left\{x \in \mathbb{R} \mid \frac{1}{2} < |x| < 1\right\}$ open in Y? Justify your answer.

(g) Consider a family of topological spaces $\{(X_\alpha, \tau_\alpha) \mid \alpha \in \Lambda\}$ and the Cartesian product $\Pi_{\alpha\in\Lambda} X_\alpha$. Define product topology on $\Pi_{\alpha\in\Lambda} X_\alpha$.

(h) Let (X, τ) be a topological space, $A \subseteq X$ and τ_A denotes the subspace topology. Prove that the inclusion map from (A, τ_A) into (X, τ) is a continuous map.

(i) Define path connected topological space with an example.

(j) Consider the subspace $A = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ in the topological space $\mathbb{R}$ with the usual topology τ_u. Prove that A is compact.

OR

[THEORY OF EQUATION]

Group-A

Answer any ***four*** questions : 12×4=48

1. (a) For what integral value $m,\ x^2+x+1$ is a factor of $x^{2m} - x^m + 1$?

(b) If $f(x)$ be a polynomial in x of degree n and α is any number real or complex, then show that

$$f(x)=f(\alpha)+f'(\alpha)(x-\alpha)+f''(\alpha)(x-\alpha)^2+......+f^n(\alpha)(x-\alpha)^n$$

(c) If x^4+px^2+qx+r has a factor of the form $(x-a)^3$, then show that $8p^3+27q^2=0$ and $p^2+12r=0$. 4+4+4=12

2. (a) A polynomial $f(x)$ leaves a remainder 10 when it is divided by $(x-2)$ and the reminder $(2x-3)$ when it is divided by $(x+1)^2$. Find the remainder when it is divided by $(x-2)(x+1)^2$.

(b) If $f(x)$ be a polynomial in x and a, b are unequal, show that remainder in the division of $f(x)$ by $(x-a)(x-b)$ is $\dfrac{(x-b)f(a)-(x-a)f(b)}{(a-b)}$.

4+4+4=12

3. (a) Solve the equation $x^4-x^3+2x^2-x+1=0$, which has four distinct roots of equal moduli.

(b) Find the conditions for which the equation $x^4-14x^3+24x+k=0$, has (i) four unequal real roots, (ii) two distinct real roots, (iii) no real root.

(c) Let $f(x)=a_0x^n+a_1x^{n-1}+.....+a_n$ where $a_0, a_1......a_n$ are integers. If $f(0), f(1)$ be both odd prove that the equation $f(x)=0$ cannot have an integer root.

4+4+4=12

4. (a) Use Sturm's theorem to find nature and position of the real roots of the equation $x^3-7x+7=0$.

(b) Prove that the solution of any reciprocal equation depends on that of a reciprocal equation of first type and of even degree.

(c) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + 4px^3 + 6qx^2 + 4rx + s = 0$ find the value of $\sum \alpha^2\beta^2(\gamma-\delta)^2$. 4+4+4=12

5. (a) Let $\alpha_1, \alpha_2 ....\alpha_n$ be the roots of the equation

$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + ... + p_n = 0$ and let $S_r = \alpha_1^r + \alpha_2^r + \alpha_3^r + ... + \alpha_n^r$, where $r > 0$, is an intreger. Then show that—

(i) $S_r + p_1S_{r-1} + ... + p_{r-1}S_1 + rp_r = 0$ if $12 \le r < n$

(ii) $S_r + p_1S_{r-1} + ... + p_nS_{r-n} = 0$ if $r > n$.

(b) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0,\ s \neq 0$ then find the values of $\sum \frac{\alpha\beta}{\gamma}, \sum \frac{\alpha^2}{\beta}$. 4+4+4=12

6. (a) Show that the equation $x^3 - 16x^2 - x - 1 = 0$ has only positive root.

(b) If equation $f(x) = 0$ has all roots real, then show that the equation $f f'' - \{f'\}^2 = 0$ has all its root imaginary.

(c) If the roots of the equation $x^3 + 3px^2 + 3qx + r = 0$ are in H.P., then show that $2q^3 = 3pqr - r^2$. 4+4+4=12

7. (a) Find the substitution of the form $x = my + n$ which will transform the following equation to a reciprocal one and hence solve it $x^4 - 7x^3 + 13x^2 - 12x + 6 = 0$.

(b) Solve the equation $x^5 - 1 = 0$, Hence find the value of $cos\frac{\pi}{5}, cos\frac{2\pi}{5}$. 8+4=12

8. (a) Solve the equation $x^7 - 1 = 0$. Deduce that $2cos\frac{2\pi}{7}, 2cos\frac{4\pi}{7}, 2cos\frac{8\pi}{7}$ are the roots of the equation $t^3 + t^2 - 2t - 1 = 0$.

(b) Solve the equation $x^3 - 13x - 35 = 0$, by taking $x = u + v$. 7+5=12

Group-B

9. Answer any ***six*** questions : 2×6=12

(a) Prove that the roots of the equation $(2x+3)(2x+4)(x-1)(4x-7)+(x+1)(2x-1)(2x-3)=0$ are all real and different.

(b) If α, β, γ, δ be the roots of the equation $x^4 - x^3 + 2x^2 + x + 1 = 0$, find the value of $(\alpha^3+1)(\beta^3+1)(\gamma^3+1)(\delta^3+1)$.

(c) Express $x^5 - 5x^4 + 12x^2 - 1$ as a polynomial in $(x-1)$.

(d) State Descartes rule of signs.

(e) Apply Descartes rule of signs to find the nature of the roots of the equation $x^6 + 7x^4 + x^2 + 2x + 1 = 0$.

(f) If α be amultiple root of order 3 of the equation $x^4 + bx^2 + cx + d = 0\,(d \neq 0)$, show that $\alpha = -\dfrac{8d}{3c}$

(g) If $(x) = x^4 - 3x^2 + 10x$, express $f(x+3)$ as a polynomial in x.

(h) Determine the multiple roots of the equation $x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2 = 0$.

(i) Solve the equation $x^4 + x^2 - 2x + 6 = 0$, it is given that 1 + *i* is a root.

(j) How many times the graph of the polynomial $(x^3-1)(x^2+x+1)$ will cross x-axis?