DEPARTMENT OF MATHEMATICS
Jhargram Raj College
JHARGRAM:: 721507

Students' Enrichment Programme

Problem of the Week

Problem Posting Date: **03.03.22** Due Date: **10.03.22**

Answer Any Two of the following:

1. Let f be an infinitely differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that, for some positive integer n,

$$f(1) = f(0) = f^{/}(0) = f^{//}(0) = \cdots = f^{n}(0) = 0.$$

Prove that $f^{n+1}(x) = 0$ for some x in $(0,1)$.

2. Assume that f is twice continuously differentiable function on $(0, \infty)$, $\lim_{x\to\infty} xf(x) = 0$, $\lim_{x\to\infty} xf^{//}(x) = 0$. Prove that $\lim_{x\to\infty} xf^{/}(x) = 0$.

3. Let f be a continuous function on $[0,1]$. Evaluate $\lim_{n\to\infty} \int_0^1 x^n f(x) dx$.

Brief Solution of the Problem Posted on 03.03.22

1. By Rolle's Theorem $\exists\ x_1 \in (0,1)$ such that $f^{/}(x_1) = 0$. Hence according to the given condition and by repeated application of Rolle's Theorem will tell us that $\exists\ x \in (0, x_n) \subset (0,1)$ such that $f^{n+1}(x) = 0$.

2. By Taylor's Theorem on $[x, x+1]$ we get

   $f(x+1) = f(x) + f^{/}(x) + \frac{1}{2} f^{//}(\lambda)$ where $\lambda \in (x, x+1)$

   Now, consider the expression

   $x f^{/}(x) = \frac{x}{x+1}(x+1) f(x+1) - x f(x) - \frac{1}{2}\frac{x}{\lambda} \lambda f^{//}(\lambda)$

   By taking the limit $x \to \infty$ both side we get the result.

3. Let $\varepsilon > 0, L = \max_{x \in [0,1]}(|f(x)| + 1)$ & $0 < \delta < \min\{\frac{\varepsilon}{2L}, 1\}$

   Observe that $| \int_{1-\delta}^{1} x^n f(x) dx | \leq \int_{1-\delta}^{1} |x|^n |f(x)| dx \leq L\delta \leq \frac{\varepsilon}{2}$

   Also $| \int_{0}^{1-\delta} x^n f(x) dx | \leq \int_{0}^{1-\delta} (1-\delta)^n |f(x)| dx \leq L\delta^{n+1}$

   So, $\lim_{n \to \infty} \int_0^1 x^n f(x) dx = 0$.