DEPARTMENT OF MATHEMATICS
Jhargram Raj College
JHARGRAM:: 721507

Students' Enrichment Programme

Problem of the Week

(SEM - VI)

Problem Posting Date: **24.03.22** Due Date: **31.03.22**

Any One:

1. Let $f_1, f_2, \dots, f_n$ be continuous real valued functions on $[a, b]$. Show that the set $\{f_1, f_2, \dots, f_n\}$ is linearly dependent on $[a, b]$ if and only if

$$\det\left(\int_a^b f_i(x) f_j(x)\,dx\right) = 0$$

2. Let α & β be real numbers such that the subgroup Γ of $(\mathbb{R}, +)$ generated by α & β is a closed set. Prove that α & β are linearly dependent over $\mathbb{Q}$.

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Brief Solution of the Problem Posted on 24.03.22

1. Claim: $0 \notin \Gamma'$
   If $0 \in \Gamma'$ and Γ contains all integer multiples of its elements so Γ is dense in $\mathbb{R}$. But then $\Gamma = \mathbb{R}$, since Γ is closed in $\mathbb{R}$. But it is not possible as Γ is countable. So $0 \notin \Gamma'$. It implies $\exists\, \gamma > 0, \gamma \in \Gamma$. If $x \in \Gamma$ & n is the largest integer such that $n\gamma \leq x$ *then* $x - n\gamma \in \Gamma$ & $0 < x - n\gamma < \gamma$ hence $x - n\gamma = 0$.

2. Let $G = G_{ij} = \int_a^b f_i(x) f_j(x) dx$. If $\det G = 0$ then G is singular. Let a be a non- zero n-vector with
   $Ga = 0 \; then \; a^T G a = 0 = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_a^b a_i f_i(x) a_j f_j(x)\, dx =$
   $\int_a^b \left(\sum_{i=1}^{n} a_i f_i(x)\right)^2 dx$
   Hence the set is linearly dependent.

   The other part is an easy calculation.