Department of Mathematics
Jhargram Raj College
Class: Mathematics(H) Sem-I

Topic: Integers **Date: 31.10.19**

1. Use the division algorithm to establish
   $(i) The\ square\ of\ any\ integer\ is\ either\ of\ the\ form\ 3k\ or\ 3k1,$
   $(ii) The\ cube\ of\ any\ integer\ has\ one\ of\ the\ form\ 9k, 9k+1, or\ 9k-1,$
   $(iii) The\ fourth\ power\ of\ any\ integer\ is\ either\ of\ the\ form\ 5k\ or\ 5k+1.$
2. Prove that $3a^2 - 1$ is never a perfect square.
3. Prove that the sum of the squares of two odd integers cannot be a perfect square.
4. Prove that $\frac{(3n)!}{(3!)^n}$ is an integer $\forall\, n \geq 0$.
5. If a is an integer not divisible by 2 or 3 then prove that $24|(a^2 + 23)$.
6. Let a,b,c be integers, no two of which are zero, and $d = \gcd(a, b, c)$. Show that $d = \gcd(\gcd(a, b), c) = \gcd(a, \gcd(b, c)) = \gcd(\gcd(a, c), b)$.
7. Find integers x,y,z satisfying $\gcd(198{,}288{,}512) = 198x + 288y + 512z$.
8. Prove that any integer of the form $8^n + 1$ is composite.
9. If $n > 1$ is an integer not of the form $6k + 3$, prove that $n^2 + 2^n$ is composite.
10. An integer is said to be square free if it is not divisible by the square of any integer greater than 1.Prove that (i) An integer(>1) is square free iff that can be factored into a product of distinct primes. (ii) Every integer(>1) is the product of a square free integer and a perfect square.
11. A positive integer n is called square-full or powerful, if $p^2|n\ \ \forall$ prime factor p of n.Show that n can be written as $n = a^2b^3, a, b\ are\ positive\ integers$.
12. Prove that the system of linear congruence's $ax + by \equiv r(mod\ n)$, $cx + dy \equiv s(mod\ n)$ has a unique solution if $\gcd(ad - bc, n) = 1$.
13. If $x \equiv a(mod\ n)$, Prove that either $x \equiv a(mod\ 2n)\ or\ \ x \equiv a + n(mod\ 2n)$.
14. The three most recent appearance of Halley's comet were in the years 1835,1910 and 1986; the next occurrence will be in 2061. Prove that $1835^{1910} + 1986^{2061} \equiv 0(mod\ 7)$.
15. Show that if $\gcd(a, n) = \gcd(a - 1, n) = 1$ then $1 + a + a^2 + \cdots + a^{\varphi(n)-1} \equiv 0(mod\ n)$.
16. Prove that the average of the positive integers less than n & relatively prime to n is $\frac{1}{2}n, n \geq 1$.
17. Show that if $f(x)$ is a polynomial with integral coefficients and if $f(a) \equiv k(mod\ m), then\ f(a + tm) \equiv k(mod\ m) \forall\ t \in \mathbb{Z}$.

References: 1. Elementary Number Theory by M.Burton.
2. An introduction to The Theory of Numbers by I.Niven,H.Zuckerman, H.Montgomery.

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