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UG/3rd Sem/STAT(H)/T/19

2019

B.Sc.

3rd Semester Examination

MATHEMATICS (Honours)

Paper - C 5-T

Theory of Real Functions and Introduction to Metric Space

Full Marks: 60

Time: 3 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Unit - 1 [Marks : 21]



1. Answer any three questions:

 $2 \times 3 = 6$

- (a) Let f: [0, 1] → [0, 1] be a continuous function in [0, 1] prove that there exist a point c in [0, 1] such that f(c) = c.
- (b) State Cauchy's criteria for the existence of $\lim_{x\to\infty} f(x)$.

[Turn Over]

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(2)

(c) Give example of functions f and g which are not continuous at a point c∈R but the sum f + g is continuous at c.



(d) What do you mean by removable discontinuity of a function at an interior point of an interval?

(e) Let
$$f(x) = \begin{cases} 2, x \in \mathbb{Q} \\ 0, x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Prove that f is discontinuous at every point c in \mathbb{R} .

2. Answer any one question:

$$5 \times 1 = 5$$

- (a) Let $D \subset R$, and f and g be functions on D to R and $g(x) \neq 0$ for all $x \in D$. Let $c \in D'$ and $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m \neq 0$. Then prove that $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{l}{m}$.
 - (b) A function $f:\mathbb{R}$ is defined by
- $f(x) = \begin{cases} x, x \in Q \\ 0, x \in \mathbb{R} Q \end{cases}$ then show that f is continuous at 0 and f has a discontinuity of the 2nd kind at every other point in \mathbb{R} .

3. Answer any one question:

 $10 \times 1 = 10$

- (a) (i) Let A, B $\subset \mathbb{R}$ and $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ be functions s.t $f(A) \subset B$. Let $C \in A$ and f is continuous at c and g is continuous at $f(c) \in B$. Then the composite function $g \circ f : A \to \mathbb{R}$ is continuous at C.
 - (ii) Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ be functions s.t $f(x) \ge 0 \quad \forall x \in D$ and f is continuous on D. Then \sqrt{f} is continuous on D. Hence prove that $h(x) = \sqrt{x^3 + 3}, x \in \mathbb{R}$ is continuous on \mathbb{R} .

3+2

(b) (i) Let I be a bounded interval and $f: I \to R$ be uniformly continuous on I. Then prove that f is

bounded on I. Show that $\cos \frac{1}{x}$ is not uniformly

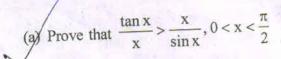
continuous.

4+2

(ii) Let $f: R \to R$ be strictly increasing and continuous and let S = f(R). Then $f^{-1}: S \to R$ is also strictly increasing and continuous.

Unit 2 [Marks: 14]

4. Answer any two questions:



- (b) Let $f: R \to R$ is defined by f(x) = |x| + |x|1, $x \in \mathbb{R}$. Find the derived function f' and specify the domain of f'.
- (c) Verify Rolle's theorem for $f(x) = \sin\left(\frac{1}{x}\right)$ on

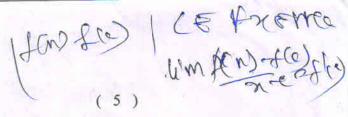
$$\left[\frac{1}{3\pi},\frac{1}{2\pi}\right].$$

5. Answer any two questions:

5×2=10

(a) Use MVT to prove the inequality $\frac{x}{\sqrt{1-x^2}} \le \sin^{-1} x < x \text{ if } 0 \le x < 1. \text{ When does}$ 4+1 the equality hold.

(b) A function f is twice differentiable on [a, b] and f(a) = f(b) = 0 and f(c) < 0 for some c in (a, b). Prove that there is at least one point ξ 5 in (a, b) for which $f''(\xi) > 0$.



(c) When a function is said to be differentiable at a point? Let I be an interval and c∈I. Let the functions $f:I \rightarrow R$ and $g:I \rightarrow R$ be differentiable at c. Then prove that f.g is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + g'(c)f(c)$$
.

Unit 3 [Marks: 14]

6. Answer any two questions:

 $2 \times 2 = 4$

- (a) State Taylor's theorem with Cauchy's form of remainder.
 - (b) Find the local extremum points of the function

$$f(x) = \frac{x^2}{(1-x)^3}$$
.



- (c) A function f is differentiable on [0, 2] and f(0) = 0, f(1) = 2, f(2) = 1. Prove that f'(c) = 0 for some c in (0, 2).
- 7. Answer any one question:

 $10 \times 1 = 10$

(a) (i) State and prove the Maclaurin's theorem with Cauchy's form of remainder.

- (ii) Use Taylor's theorem to prove that
- $1 + \frac{x}{2} \frac{x^3}{8} < \sqrt{1 + x} < 1 + \frac{x}{2}, \text{ if } x > 0. 6+4$
 - (b) (i) If $f'(x) = (x-a)^{2n} (x-b)^{2m+1}$ where m, n are positive integers, show that f has neither a maxima nor a minima at 'a' and f has a minimum at 'b'.
 - (ii) Let $f: I \to R$ be such that f has a local extrema at an interior point c of I. If f'(c) exists then prove that f'(c) = 0.

Unit 4 [Marks: 11]

8. Answer any three questions:

 $2 \times 3 = 6$

- (a) Prove that in any discrete metric space, are the sets are closed.
- (b) Let, $11^{-} \land d$ be a metric space. Then prove that $\forall A, B \in M$, $ACB \Rightarrow S(A) \leq S(B)$ where S(X) denote the diameter of a set X.
- (c) Let (M, d) be a metric space. Then prove that (M, \sqrt{d}) is also a metric space.



- (d) Let $X = \mathbb{N}$, the set of natural number, and d ye defined by $d(m, n) = \left| \frac{1}{m} \frac{1}{n} \right|$, $m, n \in \mathbb{N}$, then prove that (X, d) is a discrete metric space.
- (e) Let M be a non-empty set for x, $y \in M$, and $d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ then prove that (M, d) is a metric space.
- 9. Answer any one question:

1×5=5



- (a) State and prove that Hausdorff property.
- (b) Let $\{A_n\}$ be a sequence of open sets in R_n such that each A_n is dense in R_n . Prove that $\bigcap_{n=1}^{\infty} A_n$ is dense in R_n .